

Some Results Related to a Conjecture of Dirac's

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February 15, 2012

Abstract

We demonstrate an infinite family of pseudoline arrangements, in which an arrangement of n pseudolines has no member incident to more than $4n/9$ points of intersection. (This shows the “Strong Dirac” conjecture to be false for pseudolines.) We also prove non-trivial lower bounds on the maximum number of intersection points on any curve in an arrangement of curves in the plane, for various classes of curves. (This shows that analogs to the “Weak Dirac” theorem apply for these classes of curves.)

1 Introduction

Given an arrangement L of lines in the real projective plane, \mathbb{P}^2 , let $r(L)$ be the maximum number of vertices of L on any line of L . In 1951, G. Dirac (working in the dual context of point sets) conjectured a lower bound on $r(L)$.

Conjecture 1 (Strong Dirac). *Let L be an arrangement of n lines in \mathbb{P}^2 that do not all pass through a single point. There exists a constant c such that*

$$r(L) \geq n/2 - c.$$

In 1961, Erdős proposed a weaker version of the Strong Dirac conjecture [9]. It was proved independently in 1983 both by Beck[3] and by Szemerédi and Trotter[14], and holds for arrangements of pseudolines. An arrangement of pseudolines in the real projective plane is an arrangement of simple closed curves, any pair of which meet at a single crossing point.

Theorem 2 (Weak Dirac). *Let L be an arrangement of n pseudolines in \mathbb{P}^2 that do not all pass through a single point. Then*

$$r(L) = \Omega(n).$$

In this paper, we show that Conjecture 1 does not hold for arrangements of pseudolines, and that analogs of Theorem 2 hold for various types of arrangements.

There are relatively few combinatorial properties that are known to hold for line arrangements and not for pseudoline arrangements. The results in this paper show that the techniques used to prove the Weak Dirac theorem may be applied more generally than they have been in the past, but probably can't be used to prove the Strong Dirac conjecture.

The paper is organized as follows.

Section 2 focuses on the Strong Dirac conjecture. Traditionally, the Strong Dirac conjecture has been studied from the perspective of point sets. In this setting, the conjecture is that any set of n points includes

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a point incident to $n/2 - c$ lines spanned by the point set. There are symmetries inherent in known extremal examples, that are easier to see in the dual setting. We initially review previous results from the dual perspective used in this paper.

In Section 2.1, we describe a technique of visualizing line and pseudoline arrangements with dihedral symmetry by presenting only a single wedge, which can be used to reconstruct the entire arrangement. This method was introduced by Eppstein, on his blog [8], and further developed by Berman in an investigation of simplicial pseudoline arrangements [4].

In Section 2.2, we present an infinite family of arrangements of pseudolines, such that an arrangement of n pseudolines from this family has no member incident to more than $4n/9 - 10/9$ vertices of the arrangement. The family of pseudolines presented was previously studied by Berman [4], in the context of simplicial arrangements. This is the first time an infinite family of pseudolines has been demonstrated to violate the conclusion of the Strong Dirac conjecture.

Section 3 focuses on showing that the relationship between incidence bounds and lower bounds on the maximum number of intersection points on any line carries over to other types of arrangements. This relationship was used by both Beck [3] and by Szemerédi and Trotter [14] to prove the Weak Dirac. Szemerédi and Trotter proved an asymptotically tight lower bound on the number of incidences between a set of points and lines in the plane.

Theorem 3 (Szemerédi, Trotter [14]). *Given a set of n lines in \mathbb{P}^2 , no more than*

$$O(n^2/k^3 + n/k)$$

points are incident to k or more of the lines.

Beck proved a similar, weaker, bound. Interestingly, even though his incidence bound was not tight, Beck was able to use it to prove an asymptotically tight Weak Dirac theorem.

In Section 3.1, we formally define the class of arrangements we will consider. Informally, we consider arrangements of curves in a plane (not necessarily over an ordered field), each pair of which intersects some fixed number of times. We also prove incidence and Dirac-type bounds that follow directly from the specified combinatorics; these are analogous to results proved for line arrangements prior to 1983.

The Szemerédi-Trotter theorem is now one of many upper bounds on the number of incidences between finite sets P, L of geometric objects. In Section 3.2, we show that, for arrangements that have the combinatorial properties specified in Section 3.1, these bounds may be used to derive results analogous to the Weak Dirac theorem. In Section 3.3, we use the result of Section 3.2 to show that

- maximally intersecting families of n “well-behaved” curves, as defined by Pach and Sharir [12], include a member incident to $\Omega(n)$ points of intersection;
- families of n $(d/2)$ -flats in \mathbb{P}^d (real projective d -space), with each pair of flats intersecting at a point, include a member incident to $\Omega(n^{1-\epsilon})$ points of intersection, for any constant $\epsilon > 0$; and
- there exists a constant $\delta > 0$ such that families of n lines in $\mathbb{F}_p\mathbb{P}^2$ (finite projective plane), with $n < p$, include a member to $\Omega(n^{1/2+\delta})$ points of intersection.

2 Strong Dirac

In 1951, Dirac conjectured that among any set of n non-collinear points, P , there must exist a point incident to at least $\lceil \frac{n}{2} \rceil$ lines spanned by P [7]. This bound can be attained for odd n when the points lie on two intersecting lines. Typically, Dirac’s original conjecture is stated in a slightly weaker form (i.e., the “Strong Dirac”).¹

¹This conjecture is sometimes referred to as the *Dirac-Motzkin conjecture*, see for example the first paragraph of [3]. However, that label more often refers to the conjecture that an arrangement of n non-collinear points is spanned by at least $\frac{n}{2}$ ordinary lines.

In [1], Akiyama et al. show that the $\lfloor \frac{n}{2} \rfloor$ bound (i.e., the Strong Dirac conjecture with $c = 0$) can be attained for all sufficiently large n except those of the form $12k + 11$ (which they left as an open problem). However, there exists a family of configurations, with an arbitrarily large number of points, for which the conjecture is false for $c = 0$. This infinite family of counterexamples is due to Felsner and contains $6k + 7$ points with none incident to more than $3k + 2$ spanned lines when k is even, and $3k + 3$ when k is odd. [6, p. 313] The dual form for this family is demonstrated in Figure 1.

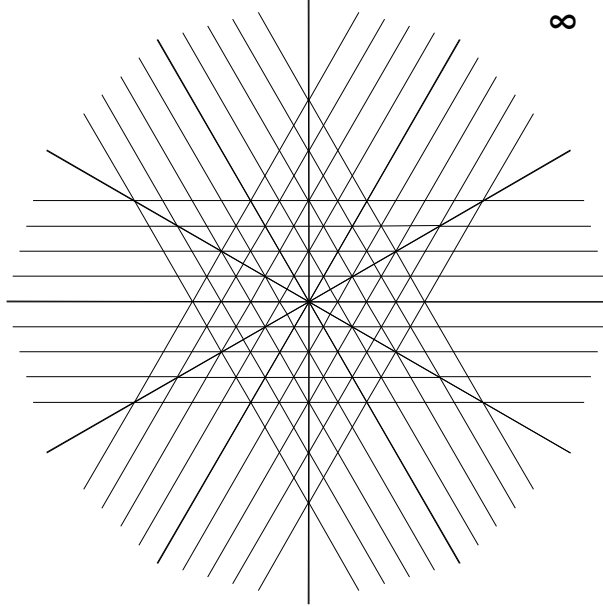


Figure 1: The dual of Felsner’s arrangement with $6k + 7 = 31$ lines (including the line at infinity) and no line incident to more than $3k + 2 = 14$ points of intersection.

No infinite family of arrangements of n such that each member has fewer than $n/2 - 3/2$ intersection points, but Grünbaum found several small arrangements with that property [10, 11]. The line arrangement $A[25,5]$ in [11] is the smallest member of the infinite family of pseudoline arrangements presented below.

2.1 Wedge presentation of symmetric pseudoline arrangements

A beautiful feature of Figure 1 is its symmetry. This drawing has the symmetry of a regular hexagon (i.e., the dihedral group D_6). While studying simplicial pseudoline arrangements (ones in which each planar face has three sides), Eppstein observed that arrangements with dihedral symmetry can be generated, similar to a kaleidoscope, from the contents of a single “wedge” [8]. Figure 2 shows a single wedge from Felsner’s arrangement.

He noted that the entire path of a line through an arrangement can be traced by considering that line to be “bouncing”, like a laser beam bouncing off mirrors, from one side of the wedge to the other. (Notice that in Figure 2 the beams must “retrace” their path after the third bounce.) In fact for straight-line arrangements, this bouncing must follow the *law of reflection*: the angle of incidence equals the angle of reflection. By applying basic trigonometry, one may deduce for straight-line arrangements the number and locations of the bounces as a function of the wedge angle and the beam’s initial angle of incidence.

To generate an arrangement from a wedge, the wedge must have an angle of π/k for some positive integer $k \geq 2$. The arrangement is produced by alternately rotating and duplicating the wedge or its mirror image, k times each, so that they fill the plane.

For pseudoline arrangements, the “bouncing” beams need not obey the law of reflection.

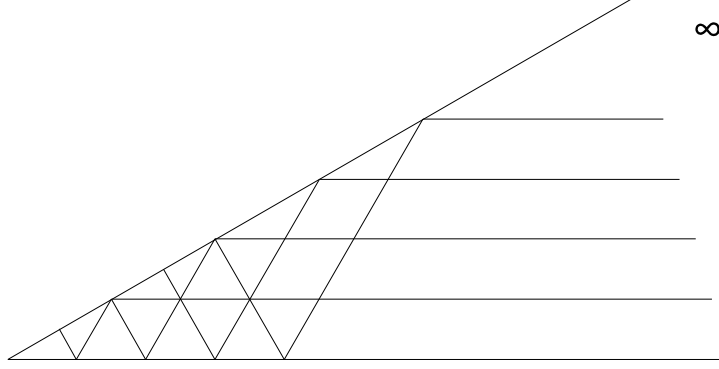


Figure 2: A single wedge from Felsner's arrangement.

As with Felsner's arrangement a beam might retrace its path after the $\lceil \frac{k}{2} \rceil^{\text{th}}$ bounce. Berman, in [4], further develops Eppstein's "kaleidoscope" method to construct and classify many types of symmetric simplicial pseudoline arrangements (including the one presented in Section 2.2).

2.2 Pseudoline counterexample to Strong Dirac

Theorem 4. *For any $j \in \mathbb{N}^+$, there exists an arrangement of $n = 18j + 7$ pseudolines such that no pseudoline is incident to more than $8j + 2$ vertices.*

Proof. We will describe the construction of a wedge for a pseudoline arrangement for arbitrary j , and show that it has the claimed number of pseudolines and intersection property. We refer to Berman [4, Fig.11] for a proof that the described wedge actually represents a pseudoline arrangement.

For an arbitrary j , the wedge angle will be $\pi/(6j + 2)$. There are four distinct symmetry classes of pseudolines, plus the line at infinity. Two of these will be represented by the sides of the wedge; we will call these the *top* and *bottom* edges. Two will be represented by beams; we will call these the *red* and *blue* beams.

Let r_i be the point at which the i^{th} bounce of the red beam occurs, counting from infinity. Likewise, let b_i be the point at which the i^{th} bounce of the blue beam occurs. After the beams reach the points r_{3j+1} or b_{3j+1} , respectively, the beams "retrace" their paths. More specifically, for any j , $r_k = r_{6j+2-k}$ and $b_k = b_{6j+2-k}$.

We call r_{3j+1} and b_{3j+1} the "terminating points" for their respective beams. Prior to reaching its terminating point, every third bounce of the *blue* beam coincides with a bounce of the *red* beam (i.e., $r_i = b_{3i}$ for $i \leq j$). The two beams are parallel to the bottom edge before the first bounce, and both b_1 and r_1 are on the top edge.

We will proceed by induction. For $j = 1$, the theorem holds; the arrangement generated from this wedge contains $3(6j + 2) + 1 = 25$ pseudolines, each of which incident to at most $8j + 2 = 10$ vertices. See Figure 3 for the wedge, and Figure 4 for the associated arrangement.

Assume that the theorem holds for $j - 1$. While maintaining for the points each existing bounce their distances to the corner of the wedge from the previous case, we reduce the wedge angle to $\pi/(6j + 2)$. In order to produce from this new wedge a valid arrangement, we must specify how to construct $\{r_{3j-1}, r_{3j}, r_{3j+1}\}$ and $\{b_{3j-1}, b_{3j}, b_{3j+1}\}$ for their respective beams.

We begin by extending the *red* by placing r_{3j-1} , r_{3j} , and r_{3j+1} on alternating sides of the wedge, each slightly closer to the corner of the wedge than the previous. This extension adds only 6 vertices to its associated *red* lines.

To extend the *blue* beam, we must cross the *red* beam placing b_{3j-1} on the opposite side of the wedge. The subsequent point, b_{3j} , coincides with r_j . Finally, place b_{3j+1} at an appropriate location on the opposite

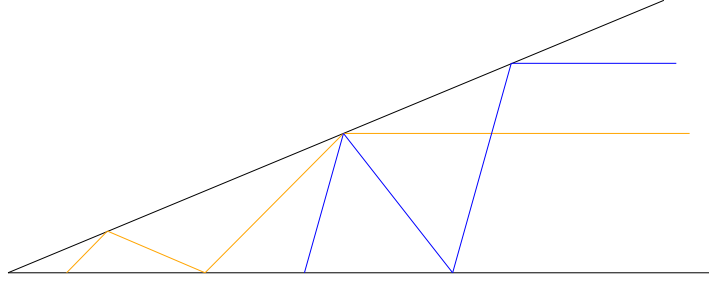


Figure 3: The wedge for $j = 1$, the base case for our induction.

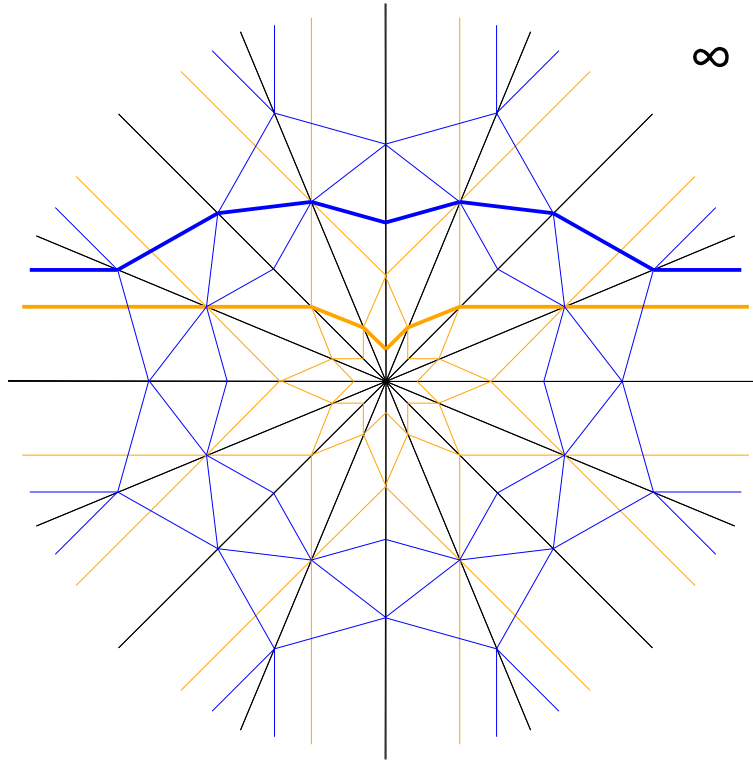


Figure 4: The arrangement for $j = 1$, containing $3(6j + 2) + 1 = 25$ pseudolines, each of which incident to at most 10 vertices.

side of the edge, slightly farther from the corner than r_{j+1} . This extension adds a total of eight vertices for the associated *blue* lines, and two more for the *red* lines.

We must now consider the additional vertices formed on the sides of the wedge (which correspond to the axes of symmetry). To one set of axes, we added eight vertices each; to the other, we added only six each. See Figure 5 for the $j = 2$ case, i.e., the first complete “extension”.

In the resulting arrangement, there will be $18j + 7$ lines with none incident to more than $8(j - 1) + 2 + 8 = 8j + 2$ vertices, completing the induction. \square

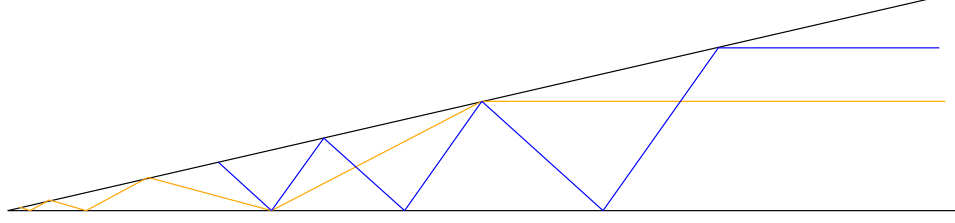


Figure 5: The wedge for $j = 2$.

3 Weak Dirac

In this section, we show that incidence bounds imply analogs to the Weak Dirac theorem for a variety of different types of arrangements.

3.1 α -Curve combinatorics

First, we will define exactly the class of arrangements we will consider.

Definition 5. Let $G = (L, P, I)$ be a connected, simple, bipartite graph on vertex sets L and P and edge set I . If each pair of elements in L is connected to exactly α elements of P , then G is an α -curve combinatorics.

This is a generalization of a line combinatorics, as defined by Bartolo et. al. [2]. It can be thought of as the unordered combinatorics of a set of curves in the plane, with each pair of curves intersecting α times.

Definition 6. For any curve combinatorics $G = (L, P, I)$, let $t_k(G)$ be the number of elements of P with degree k and, let $r(G)$ be the maximum degree of any element in L .

Some bounds on t_k and r follow from the definition of α -curve combinatorics.

The following upper bounds on t_k are a generalization of Lemma 2.1 in [3].

Theorem 7. Let $G = (L, P, I)$ be an α -curve combinatorics with $|L| = n$.

1. $t_k(G) \leq \alpha \binom{n}{2} / \binom{k}{2}$ for $2 \leq k \leq n$;
2. $t_k(G) < 2\alpha n/k$ for $\alpha(2n)^{1/2} \leq k \leq n$.

Proof.

1. We will count triplets (l, l', p) , with $l, l' \in L$, $p \in P$, and $(l, p), (l', p) \in I$, in two different ways.

There are $\binom{n}{2}$ pairs of elements in L ; each of these pairs is adjacent to α elements in P . Thus, there are $\alpha \binom{n}{2}$ such triplets.

There are at least $t_k(G)$ elements of P having degree k or more; each of these is adjacent to $\binom{k}{2}$ or more pairs of lines. Thus, the number of such triplets is at least $t_k(G) \binom{k}{2}$. Solving the inequality yields the result.

2. Let $P = \{p_1, p_2, \dots, p_t\}$, and let $L_i \subseteq L$ be the elements of L that are adjacent to p_i .

$$\begin{aligned} n &= \left| \bigcup_{i=1}^t L_i \right| \\ &\geq \sum_{i=1}^t |L_i| - \sum_{1 \leq i < j \leq t} |L_i \cap L_j| \\ &\geq kt - \alpha \binom{t}{2}. \end{aligned}$$

Assume, for contradiction, that $t \geq \lceil 2\alpha n/k \rceil$.

$$n \geq 2\alpha n - \alpha \binom{(2n)^{1/2}}{2} > n.$$

□

Dirac's original proof of the $\Omega(n^{1/2})$ bound on r relied only on combinatorial arguments [7]. The following is a generalization of his argument.

Theorem 8. *Let $G = (L, P, I)$ be a α -curve combinatorics with $|L| = n$ such that no set of α elements of P is adjacent to every member of L . Then*

$$r(G) = \Omega(n^{1/(\alpha+1)}).$$

Proof. Let g be the maximum degree of any element of L . Let h be the maximum number of elements of L adjacent to any subset $P_\alpha \subset P$ of size α .

We will show that $g \geq h$. Let $P' \subset P$ be a subset of size α with each member adjacent to the same set $L' \subset L$ of size h . By our hypothesis, there exists a $l' \in L$ not adjacent to every element of P' . By the definition of an α -curve combinatorics, there must be α paths of length 2 between l' and each element of L' . However, no two elements of L' can both be adjacent to the same element $p' \in (P \setminus P')$, otherwise there will be a $K_{2, \alpha+1}$. Thus, l' must be adjacent to at least h elements of P , and thus $g \geq h$.

Assume $h = n^x$, for some value x . Let $l \in L$ have degree g . Since l has α paths of length 2 to each $l' \in L \setminus l$, $\binom{g}{\alpha} h \geq n - 1$, which implies $g \geq n^{(1-x)/\alpha}$.

We've demonstrated that $g \geq \max(n^x, n^{(1-x)/\alpha})$ which has its minimum value when $x = 1/(\alpha + 1)$. This completes the proof. □

3.2 A conditional Weak Dirac for α -curve combinatorics

The following lemma can be used to easily prove analogs to the Weak Dirac theorem for several specific types of arrangements.

Lemma 9. *Let δ, ϵ, ζ be constants with $0 \leq \delta < \epsilon/2$, and $0 \leq \zeta < 1/2$. Let $G = (L, P, I)$ be an α -curve combinatorics with $|L| = n$. If*

$$t_k(G) = O(n^{2+\delta}/k^{2+\epsilon} + n/k)$$

for all k such that $\Omega(n^\zeta) \leq k < n$, then either

1. G is a $K_{n, \alpha}$, or
2. there is an element $\ell \in L$ with degree $\Omega(n^{1-\gamma})$,

where $\gamma = \max(\delta/\epsilon, \zeta)$.

Proof. For any pair $\{l', l''\} \in L \times L$, $l' \neq l''$, let $P_{\{l', l''\}} \subset P$ be the elements to which both l' and l'' are adjacent. Note that $|P_{\{l', l''\}}| = \alpha$ for all pairs $\{l', l''\}$. Let l_d be the number of pairs $\{l', l''\}$ such that $\min(\deg(p) : p \in P_{\{l', l''\}}) = d$. Counting pairs of elements in L two ways,

$$\sum_{i=1}^n l_i = \binom{n}{2}. \quad (1)$$

Proposition 10. *For any $c < 1$, there exists a v such that*

$$\sum_{i=2^v n^\gamma}^{n/2^v} l_i < c \binom{n}{2}.$$

Proof. Let $2^j = \Omega(n^\zeta)$. The total number of points with degree 2^j (i.e., t_{2^j}) is bounded above by $C(n^{2+\delta}/2^{j(2+\epsilon)} + n/2^j)$ for some constant C . In addition, an element of P with degree 2^{j+1} is adjacent to at most $\binom{2^{j+1}}{2}$ pairs of elements in L . Thus,

$$\begin{aligned} \sum_{i=2^j}^{2^{j+1}} l_i &< \frac{1}{\alpha} \binom{2^{j+1}}{2} t_{2^j} \\ &< (2^{2j+1} C / \alpha) (n^{2+\delta} / 2^{j(2+\epsilon)} + n / 2^j) \\ &< (2C / \alpha) (n^{2+\delta} / 2^{\epsilon j} + n 2^j). \end{aligned}$$

Let $C_t = 2C / \alpha$.

$$\begin{aligned} \sum_{i=2^v n^\gamma}^{n/2^v} l_i &< \sum_{j=v+\gamma \lfloor \log_2 n \rfloor}^{\lfloor \log_2 n \rfloor - v} \left(\sum_{i=2^j}^{2^{j+1}} l_i \right) \\ &< \sum_{j=v+\gamma \lfloor \log_2 n \rfloor}^{\lfloor \log_2 n \rfloor - v} C_t (n^{2+\delta} / 2^{\epsilon j} + n 2^j) \\ &< C_t \left(n^{2+\delta} \sum_{j=v+\gamma \lfloor \log_2 n \rfloor}^{\infty} 2^{-\epsilon j} + n \sum_0^{\lfloor \log(n) \rfloor - v} 2^j \right) \\ &< C_t \left(n^{2+\delta} (2^{-\epsilon(v+\gamma \lfloor \log_2 n \rfloor)}) / (1 - 2^{-\epsilon}) + n^2 / 2^{v-1} \right) \end{aligned}$$

Since $\delta/\epsilon \leq \gamma$,

$$\begin{aligned} &< C_t \left(n^{2+\delta} 2^{-\epsilon v} n^{-\epsilon(\delta/\epsilon)} / (1 - 2^{-\epsilon}) + n^2 / 2^{v-1} \right) \\ &< C_t \left(2^{-\epsilon v} n^{2+\delta} n^{-\delta} / (1 - 2^{-\epsilon}) + (1/2^{v-1}) n^2 \right) \\ &< C_t \left(\frac{1}{2^{\epsilon(v-1)}(2^\epsilon - 1)} + \frac{1}{2^{v-1}} \right) n^2. \end{aligned}$$

Choosing a sufficiently large v ensures that the sum is less than $c \binom{n}{2}$, completing the proof of Proposition 10. \square

Proposition 11. *One of the following must hold:*

1. *either G contains a complete bipartite subgraph $K_{\Omega(n), \alpha}$, or*
2. $|P| = \Omega(n^{2-\gamma})$.

Proof. Proposition 10 together with equation 1 immediately implies that either

1. $\sum_{i=1}^{2^v n^\gamma} l_i = \Omega(n^2)$, or
2. $\sum_{i=n/2^v}^n l_i = \Omega(n^2)$.

If $\sum_{i=1}^{2^v n^\gamma} l_i = \Omega(n^2)$, then $|P| = \Omega(n^2)/(2^v n^\gamma) = \Omega(n^{2-\gamma})$, and the theorem is proved.

Assume $\sum_{i=n/2^v}^n l_i = \Omega(n^2)$. We will construct induced subgraphs $G \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_\alpha$, as needed, by incrementally removing nodes from G .

Select a point $p_1 \in P$ with $\deg(p_1) \geq n/2^v$. Let $L_1 \subseteq L$ be the set of nodes adjacent to p_1 . Let P_1 be the set of elements in $P \setminus \{p_1\}$ that are adjacent to some element in L_1 . Construct $G_1 = (L_1, P_1, I_1)$ as the induced subgraph formed by L_1 and P_1 . Clearly, G_1 is a $(\alpha - 1)$ -curve combinatorics.

From Proposition 10, either $|P_1| = \Omega(|L_1|^{2-\gamma}) = \Omega(n^{2-\gamma})$, or there is an element $p_2 \in P$ with $\deg(p_2) \geq n/2^{2v}$. In the second case, repeat the above procedure to get a graph G_2 .

This process may be repeated until either:

- we find a subset $P_j \subseteq P$, $j \leq \alpha$, with $|P_j| = \Omega((n/2^{jv})^{2-\gamma}) = \Omega(n^{2-\gamma})$, or
- we can construct a complete bipartite subgraph of G containing the nodes $L_\alpha \cup \{p_1\} \cup \dots \cup \{p_\alpha\}$, with $|L| \geq n/2^{v\alpha} = \Omega(n)$.

This completes the proof of Proposition 11. □

By Proposition 11, there are two cases to consider.

First, assume $|P| = \Omega(n^{2-\gamma})$. Since every point of P is adjacent to at least one line of L , the pigeonhole principle implies that there must be a line adjacent to $\Omega(n^{1-\gamma})$ points of P , proving the theorem.

Now, assume there exists a complete bipartite subgraph $G' = (L', P', I')$, $G' \subseteq G$ with $|L'| = \Omega(n)$ and $|P'| = \alpha$. If $|L'| = n$, then G is a $K_{n,\alpha}$, proving the theorem. Otherwise, there exists an element $l \in (L \setminus L')$ not adjacent to all α elements of P' . By the definition of a curve combinatorics, l must have α paths of length 2 to every element of L' . However, since no two elements of L' can both be incident to the same point in $P \setminus P'$ (otherwise a forbidden subgraph would exist), l must be adjacent to $|L'| = \Omega(n)$ elements of $P \setminus P'$. □

3.3 Applications

3.3.1 Pach-Sharir curves

Pach and Sharir defined “well-behaved” curves as follows [12].

Let C be a set of simple curves in the plane. The set C is said to have β degrees of freedom and multiplicity-type α , if

- for β points there are at most α curves passing through all of them, and
- any pair of curves intersect in at most α points.

These two conditions correspond, respectively, to a forbidden $K_{\beta+1,\alpha}$ and a forbidden $K_{2,\alpha+1}$ in the bipartite graph encoding the incidences between such a set of curves and a set of points.

Pach and Sharir proved the following upper bound on the number of incidences between “well-behaved” curves and points [12].

Theorem 12. *Let P be a set of m points and let C be a set of n simple curves all lying in the plane. If C has β degrees of freedom and multiplicity-type α , then the number of incidences between P and C is*

$$O_{\alpha,\beta} \left(m^{\beta/(2\beta-1)} n^{(2\beta-2)/(2\beta-1)} + m + n \right).$$

We can use this with Lemma 9 to prove the following corollary.

Corollary 13. *Let C be an arrangement of n simple curves all lying in the plane. If C has β degrees of freedom, multiplicity-type α , and every pair of curves intersects exactly α times, then either all of the curves intersect in a single set of α points, or*

$$r(C) = \Omega_{\alpha,\beta}(n).$$

Proof. Let $t_k = t_k(C)$. The corollary will follow immediately from Lemma 9 and the following upper bound on the number of curves incident to k or more points:

$$t_k = O(n^2/k^{2+\epsilon} + n/k).$$

Since there are at most $\alpha \binom{n}{2}$ points, this inequality clearly holds for small k . Otherwise, by Theorem 12 we have

$$kt_k = O(t_k^{\beta/(2\beta-1)} n^{(2\beta-2)/(2\beta-1)} + t_k + n).$$

Thus, either $t_k = O(n/k)$ or $t_k = O(n^2/k^{2+\epsilon})$, where $\epsilon = 1/(\alpha - 1)$. □

3.3.2 $(d/2)$ -Flats in \mathbb{R}^d

Solymosi and Tao [13] proved an incidence bound for $d/2$ dimensional algebraic varieties in \mathbb{R}^d under some pseudoline-like conditions. Their theorem can be used to prove a corresponding bound on r for such varieties. Rather than stating their result in its full generality, we'll consider only the special case of p -flats in \mathbb{R}^d , with $d \geq 2p$.

Theorem 14. *Let $\epsilon > 0$, $p \geq 1$, and $d \geq 2p$. Let P be a set of m points, and let L be a set of n p -flats in \mathbb{R}^d such that any two distinct flats in L intersect in at most one point. Then, the number of incidences between L and P is*

$$O_{p,\epsilon} \left(m^{2/3+\epsilon} n^{2/3} + m + n \right).$$

Corollary 15. *Let $\epsilon > 0$, $p \geq 1$ and $d \geq 2p$. Let L be an arrangement of n p -flats in \mathbb{R}^d such that any two distinct flats in L intersect in exactly one point. Either all of the flats in L intersect in a single point, or*

$$r(L) = \Omega_{p,\epsilon}(n^{1-\epsilon}).$$

Proof. Let $t_k = t_k(L)$. The corollary will follow immediately from Lemma 9 and the following upper bound on the number of p -flats incident to k or more points:

$$t_k = O_{p,\epsilon}(n^{2+\epsilon'}/k^{3+\epsilon''} + n/k).$$

Since there are at most $\binom{n}{2}$ points, this inequality holds for small k . From Theorem 14,

$$kt_k = O_{p,\epsilon}(t_k^{2/3+\epsilon} n^{2/3} + n + t_k).$$

Thus, either $t_k^{1-3\epsilon} \leq n^2/k^3$ or $t_k \leq n/k$. □

3.3.3 Finite planes without too many lines

We can also use the following theorem of Bourgain, Katz, and Tao [5].

Let $\mathbb{F}_p\mathbb{P}^2$ be the projective finite plane over the field with p elements, for p prime.

Theorem 16 (Bourgain, Katz, Tao). *Let P and L be points and lines in $\mathbb{F}_p\mathbb{P}^2$ with cardinality $|P|, |L| \leq N \leq p$. Then the number of incidences between L and P is*

$$O(N^{3/2-\epsilon}),$$

for some universal constant $\epsilon > 0$.

This gives the following corollary:

Corollary 17. *Let L be an arrangement of lines in $\mathbb{F}_p\mathbb{P}^2$ with cardinality $|L| = n$. If $n \leq p$ and $\sum_{k \geq n^{1/2-\epsilon/4}} t_k \leq p$, then either all the lines intersect in a single point, or*

$$r(L) = \Omega(n^{1/2+\epsilon/4}),$$

where ϵ is the same constant as in Theorem 16.

Proof. Let $t_k = t_k(L)$.

Assume $k \geq n^{(1-\epsilon)/(2-\epsilon)}$. By Theorem 7, $t_k = O(n^2/k^2)$, so, under this assumption, $n \geq t_k^{(2-\epsilon)/2}$.

By Theorem 7, if $k \geq (2n)^{1/2}$, then $t_k < 2n/k$. On the other hand, if $k = O(n^{1/2})$, then t_k is maximized when $t_k = \Omega(n)$.

Assume $k = O(n^{1/2})$, $n = O(t_k)$. By Theorem 16,

$$kt_k = O\left(t_k^{3/2-\epsilon}\right).$$

Using the assumption that $n \geq t_k^{(2-\epsilon)/2}$,

$$\begin{aligned} t_k &= O\left(t_k^{1/2+\epsilon} t_k^{1-2\epsilon}/k\right) \\ &= O\left(t_k^{1/2+\epsilon} n^{2(1-2\epsilon)/(2-\epsilon)}/k\right) \\ t_k^{(1-2\epsilon)/2} &= O\left(n^{2(1-2\epsilon)/(2-\epsilon)}/k\right) \\ t_k &= O\left(n^{4/(2-\epsilon)}/k^{2/(1-2\epsilon)}\right) \\ &= O\left(n^{2+2\epsilon/(2-\epsilon)}/k^{2+4\epsilon}\right). \end{aligned}$$

Any example showing that the Szemerédi-Trotter theorem is tight can be constructed in a sufficiently large finite field, so $\epsilon \leq 1/6$.

$$t_k = O\left(n^{2+12\epsilon/11}/k^{2+4\epsilon}\right).$$

Removing the assumption that $k = O(n^{1/2})$,

$$t_k = O\left(n^{2+12\epsilon/11}/k^{2+4\epsilon} + n/k\right).$$

Since we've assumed that $k \geq n^{(1-\epsilon)/(2-\epsilon)}$, Lemma 9 implies that there is an element of L incident to $\Omega(n^{1-(1-\epsilon)/(2-\epsilon)}) = \Omega(n^{1/2+\epsilon/4})$ points. \square

4 Conclusion

The most surprising result presented here is that the Strong Dirac conjecture does not hold for pseudoline arrangements. This immediately raises a number of interesting questions.

There is no reason to expect that $4/9$ is the best possible constant in the Weak Dirac for pseudolines, and the gap between $4/9$ and the best known lower bound is quite large.

Problem 1. *What is the supremum of values c for which*

$$r(L) \geq cn + o(n)$$

for all pseudoline arrangements L ?

A significant question is whether (and by how much) the bound on $r(L)$ for line arrangements differs from that for pseudoline arrangements.

Problem 2. *Is it possible to prove a lower bound on $r(L)$ that holds for line arrangements and not for pseudoline arrangements?*

One feature of the family of pseudoline arrangements presented in Section 2.2 is that $(n-1)/3$ lines are all incident to a single vertex. A natural question is whether this is an essential feature of any pseudoline counterexample to the Strong Dirac².

Problem 3. *Is there an infinite family of arrangements of n pseudolines, such that*

- *no vertex of any arrangement in the family is incident to $\Omega(n)$ pseudolines, and*
- *no member of any arrangement is incident to more than $n/(2+\epsilon)$ pseudolines for some $\epsilon > 0$?*

Both Felsner's example, and the example presented in Section 2.2 have a high degree of symmetry. Assuming that the Strong Dirac holds for line arrangements, it may be easier to prove for the special case of symmetric line arrangements.

Problem 4. *Does the Strong Dirac hold for line arrangements with a non-trivial symmetry group? For line arrangements with dihedral symmetry?*

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²Thank you to N. Linial for pointing this out.

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